Pricing of Asian temperature risk

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Abstract

Weather derivatives (WD) are different from most financial derivatives because the underlying weather cannot be traded and therefore cannot be replicated by other financial instruments. The market price of risk (MPR) is an important parameter of the associated equivalent martingale measures used to price and hedge weather futures/options in the market. The majority of papers so far have priced non-tradable assets assuming zero MPR, but this assumption underestimates WD prices. We study the MPR structure as a time dependent object with concentration on emerging markets in Asia. We find that Asian Temperatures (Tokyo, Osaka, Beijing, Teipei) are normal in the sense that the driving stochastics are close to a Wiener Process. The regression residuals of the temperature show a clear seasonal variation and the volatility term structure of CAT temperature futures presents a modified Samuelson effect. In order to achieve normality in standardized residuals, the seasonal variation is calibrated with a combination of a fourier truncated series with a GARCH model and with a local linear regression. By calibrating model prices, we implied the MPR from Cumulative total of 24-hour average temperature futures (C24AT) for Japanese Cities, or by knowing the formal dependence of MPR on seasonal variation, we price derivatives for Kaohsiung, where weather derivative market does not exist. The findings support theoretical results of reverse relation between MPR and seasonal variation of temperature process.

Keywords: Weather derivatives, continuous autoregressive model, CAT, CDD, HDD, market price of risk, risk premium
JEL classification: G19, G29, G22, N23, N53, Q59

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Notation of abbreviations

cdf	Cumulative distribution function
e.g.	Exempli gratia; for example
et al.	among others
i.e.	id est.; that is
T	Expiration time
K	Strike price
τ	Threshold time event
τ_1	Start of measurement period
τ_1	End of measurement period
$(\Omega, \mathcal{F}, \mathcal{F}_t, P)$	Probability space
$\phi(x)$	The standard normal pdf
$\Phi(x)$ The standard normal cdf	
CME	Chicago MErcantile Exchange
CDD	Cooling Degree Days
HDD	Heating Degree Days
CAT	Cumulative Average Temperature
C24AT	accumulated total of 24-hour average temperature
I_p	$p \times p$ Identity matrix

1 Introduction

Global warming increases weather risk by rising temperatures and increasing between weather patterns. PricewaterhouseCoopers (2005) releases the top 5 sectors in need of financial instruments to hedge weather risk. An increasing number of business hedge risks with weather derivatives (WD): financial contracts whose payments are dependent on weather-related measurements.

Chicago Mercantile Exchange (CME) offers monthly and seasonal futures and options contracts on temperature, frost, snowfall or hurricane indices in 24 cities in the US., six in Canada, 10 in Europe, two in Asia-Pacific and three cities in Australia. Notional value of CME Weather products has grown from 2.2 USD billion in 2004 to 218 USD billion in 2007, with volume of nearly a million contracts traded, CME (2005). More than the half of the total weather derivative business comes from the energy sector, followed by the construction, the retail and the agriculture industry, according to the Weather Risk Management Association, PricewaterhouseCoopers (2005). The use of weather derivatives can be expected to grow further.

Weather derivatives are different from most financial derivatives because the underlying weather cannot be traded and therefore cannot be replicated by other financial instruments. The pricing of weather derivatives has attracted the attention of many researchers. Dornier and Querel (2000) and Alaton, Djehiche and Stillberger (2002) fitted Ornstein-Uhlenbeck stochastic processes to temperature data and investigated future prices on temperature indices. Campbell and Diebold (2005) analyse heteroscedasticity in temperature volatily and Benth (2003), Benth and Saltyte Benth (2005) and Benth, Saltyte Benth and Koekebakker (2007) develop the non-arbitrage framework for pricing different temperature derivatives prices.

The market price of risk (MPR) is an important parameter of the associated equivalent martingale measures used to price and hedge weather futures/options in the market. The majority of papers so far have priced non-tradable assets assuming zero MPR, but this assumption underestimates WD prices. The estimate of the MPR is interesting by its own and has not been studied earlier. We study therefore the MPR structure as a time dependent object with concentration on emerging markets in Asia. We find that Asian Temperatures (Tokyo, Osaka, Beijing, Teipei and Koahsiung) are normal in the sense that the driving stochastics are close to a Wiener Process. The regression residuals of the temperature show a clear seasonal variation and the volatility term structure of CAT temperature futures presents a modified Samuelson effect. In order to achieve normality in standardized residuals, the seasonal dependence of variance of residuals is calibrated with a truncanted Fourier function and a Generalized Autoregressive Conditional Heteroscedasticity GARCH(p,q). Alternatively, the seasonal variation is smoothed with a Local Linear Regression estimator, that it is based on a locally fitting a line rather than a constant. By calibrating model prices, we imply the market price of temperature risk for Asian futures. Mathematically speaking this is an inverse problem that yields in estimates of MPR. We find that the market price of risk is different from zero when it is assumed to be (non)-time dependent for different contract types and it shows a seasonal structure related to the seasonal variance of the temperature process. The findings support theoretical results of reverse relation between MPR and seasonal variation of temperature process, indicating that a simple parametrization of the MPR is possible and therefore, it can be infered by calibration of the data or by knowing the formal dependence of MPR on seasonal variation for regions where there is not weather derivative market.

This chapter is structured as follows, the next section we discuss the fundamentals of temperature derivatives (future and options), their indices and we also describe the monthly temperature futures traded at CME, the biggest market offering this kind of product. Section 3, - the econometric part - is devoted to explaining the dynamics of temperature data by using a continuous autoregressive model (CAR). In section 4, - the financial mathematics part - the weather dynamics are connected with pricing. In section 5, the dynamics of Tokyo and Osaka temperature are studied and by using the implied MPR from cumulative total of 24-hour average temperature futures (C24AT) for Japanese Cities or by knowing the formal dependence of MPR on seasonal variation, new derivatives are priced, like C24AT temperatures in Kaohsiung, where there is still no formal weather derivative market. Section 6 concludes the chapter. All computations in this chapter are carried out in Matlab version 7.6. The temperature data and the Weather Derivative data was provided by Bloomberg Professional service.

2 The temperature derivative market

The largest portion of futures and options written on temperature indices is traded on the CME, while a huge part of the market beyond these indices takes place OTC. A call option is a contract that gives the owner the right to buy the underlying asset for a fixed price at an agreed time. The owner is not obliged to buy, but exercises the option only if this is of his or her advantage. The fixed price in the option is called the strike price, whereas the agreed time for using the option is called the exercise time of the contract. A put option gives the owner the right to sell the underlying. The owner of a call option written on futures $F_{(\tau,\tau_1,\tau_2)}$ with exercise time $\tau \leq \tau_1$ and measurement period $[\tau_1, \tau_2]$ will receive:

$$\max\left\{F_{(\tau,\tau_1,\tau_2)} - K, 0\right\} \tag{1}$$

where K is the strike price. Most trading in weather markets centers on temperature hedging using either heating degree days (HDD), cooling degree days (CDD) and Cumulative Averages (CAT). The HDD index measures the temperature over a period $[\tau_1, \tau_2]$, usually between October to April, and it is defined as:

$$\mathrm{HDD}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \max(c - T_u, 0)du \tag{2}$$

where c is the baseline temperature (typically 18°C or 65°F) and T_u is the average temperature on day u. Similarly, the CDD index measures the temperature over a period $[\tau_1, \tau_2]$, usually between April to October, and it is defined as:

$$\mathrm{CDD}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \max(T_u - c, 0)du \tag{3}$$

The HDD and the CDD index are used to trade futures and options in 20 US cities (Cincinnati, Colorado Springs, Dallas, Des Moines, Detroit, Houston, Jacksonville, Kansas City, Las Vegas, Little Rock, Los Angeles, Minneapolis-St. Paul, New York, Philidelphia, Portland, Raleigh, Sacramento, Salt Lake City, Tucson, Washington D.C), six Canadian cities (Calgary, Edmonton, Montreal, Toronto, Vancouver and Winnipeg) and three Australian cities (Brisbane, Melbourne and Sydney).

The CAT index accounts the accumulated average temperature over a period $[\tau_1, \tau_2]$ days:

$$\mathrm{CAT}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} T_u du \tag{4}$$

where $T_u = \frac{T_{t,max} - T_{t,min}}{2}$. The CAT index is the substitution of the CDD index for nine European cities (Amsterdam, Essen, Paris, Barcelona, London, Rome, Berlin, Madrid, Oslo, Stockholm). Since $\max(T_u - c, 0) - \max(c - T_u, 0) = T_u - c$, we get the HDD-CDD parity

$$\mathrm{CDD}(\tau_1, \tau_2) - \mathrm{HDD}(\tau_1, \tau_2) = \mathrm{CAT}(\tau_1, \tau_2) - c(\tau_2 - \tau_1) \tag{5}$$

Code	First-trade	Last-trade	τ_1	τ_2	CME	$C\hat{24}AT$
F9	20080203	20090202	20090101	20090131	200.2	181.0
G9	20080303	20090302	20090201	20090228	220.8	215.0
H9	20080403	20090402	20090301	20090331	301.9	298.0
J9	20080503	20100502	20090401	20090430	460.0	464.0
K9	20080603	20090602	20090501	20090531	592.0	621.0

Table 1: C24AT Contracts listed for Osaka at the beginning of the measurement period ($\tau_1 - \tau_2$) and CME and C24ATs from temperature data. Source: Bloomberg

Therefore, it is sufficient to analyse only HDD and CAT indices. An index similar to the CAT index is the Pacific Rim Index, which measures the accumulated total of 24-hour average temperature (C24AT) over a period $[\tau_1, \tau_2]$ days for Japanese Cities (Tokyo and Osaka):

$$\text{C24AT}(\tau_1, \tau_2) = \int_{\tau_1}^{\tau_2} \widetilde{T}_u du \tag{6}$$

where $\widetilde{T}_u = \frac{1}{24} \int_1^{24} T_{u_i} du_i$ and T_{u_i} denotes the temperature of hour u_i.

The options at CME are cash settled i.e. the owner of a future receives 20 times the Degree Day Index at the end of the measurement period, in return for a fixed price (the future price of the contract). The currency is British pounds for the European Futures contracts, US dollars for the US contracts and Japanese Yen for the Asian cities. The minimum price increment is one Degree Day Index point. The degree day metric is Celsius and the termination of the trading is two calendar days following the expiration of the contract month. The Settlement is based on the relevant Degree Day index on the first exchange business day at least two calendar days after the futures contract month. The accumulation period of each CAT/CDD/HDD/C24AT index futures contract begins with the first calendar day of the contract month and ends with the calendar day of the contract month. Earth Satellite Corporation reports to CME the daily average temperature. Traders bet that the temperature will not exceed the estimates from Earth Satellite Corporation.

At the CME, the measurement periods for the different temperature indices are standarized to be each month of the year and two seasons: the winter (October - April) and summer season (April - October). The notation for temperature futures contracts is the following: F for January, G for February, H for March, J for April, K for May, M for June, N for July, Q for August, U for October, V for November and X for December. J7 stands for 2007, J8 for 2008, etc. Table 1 describes the CME future data for Osaka historical temperature data, obtained from Earth Satellite (EarthSat) corporation (the providers of temperature derivative products traded at CME). The J9 contract corresponds to a contract with the temperature measurement period from 20090401 (τ_1) to 20090430 (τ_2) and trading period (t) from 20080503 to 20080502. At trading day t, CME issues seven contracts ($i = 1, \cdots, 7$) with measurement period $\tau_1^i \leq t < \tau_2^i$ (usually with $i = 1$) or $t \leq \tau_1^i < \tau_2^i$ with $i = 1, \dots, 7$ (six months ahead from the trading day t). Table 1 also shows the C24AT from the historical temperature data obtained from Osaka Kansai International Airport. Both indices are notably differed and the raised question here is related to weather modelling and forecasting.

The fair price of a temperature option contract, derived via a hedging strategy and the principle of no arbitrage, requires a stochastic model for the temperature dynamics. In the next section, a continuous-time process AR(p) ($CAR(p)$) is proposed for the temperature modelling.

3 Temperature Dynamics

Suppose that $(\Omega, \mathcal{F}, P)$ is a probability space with a filtration $\{\mathcal{F}_t\}_{0\leq t\leq\tau_{\max}}$, where $\tau_{\max}$ denotes a maximal time covering all times of interest in the market. The various temperature forward prices at time t depends explicitly on the state vector $\mathbf{X}_t$. Let $X_{q(t)}$ be the q'th coordinate of the vector $\mathbf{X}_t$ with $q = 1, .., p$. Here it is claimed that $\mathbf{X}_t$ is namely the temperature at times t, $t-1$, $t-2$, $t-3\ldots$. Following this nomenclature, the temperature time series at time t ($q = 1$):

$$T_t = \Lambda_t + X_{1(t)} \tag{7}$$

with Λ_t a deterministic seasonal function. $X_{q(t)}$ can be seen as a discretization of a continuous-time process AR(p) ($CAR(p)$). Define a $p \times p$-matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & & \ddots & & 0 & \vdots \\ 0 & \ldots & \ldots & 0 & 0 & 1 \\ -\alpha_p & -\alpha_{p-1} & \ldots & & 0 & -\alpha_1 \end{pmatrix} \tag{8}$$

in the vectorial Ornstein-Uhlenbleck process $\mathbf{X}_t \in \mathbb{R}^p$ for $p \geq 1$ as:

$$d\mathbf{X}_t = A\mathbf{X}_t dt + \mathbf{e}_{pt}\sigma_t dB_t \tag{9}$$

where $\mathbf{e}_k$ denotes the k'th unit vector in $\mathbb{R}^p$ for $k = 1, ...p$, $\sigma_t > 0$ states the temperature volatility, B_t is a Wiener Process and α_k are positive constants. Note that the form of the $A_{p\times p}$-matrix, makes $X_{q(t)}$ to be a Markov process.

By applying the multidimensional *Itô Formula*, the process in Equation (9) has the explicit form:

$$\mathbf{X}_s = \exp\{A(s-t)\}\mathbf{x} + \int_t^s \exp\{A(s-u)\}\mathbf{e}_p\sigma_u dB_u \tag{10}$$

for $s \geq t \geq 0$ and stationarity holds when the eigenvalues of A have negative real part or the variance matrix $\int_0^t \sigma_{t-s}^2 \exp\{A(s)\}\mathbf{e}_p\mathbf{e}_p^\top \exp\left\{A^\top(s)\right\}ds$ converges as $t \to \infty$.

There is an analytical link between $X_{q(t)}$, and the lagged temperatures up to time $t-p$. We first say that the state vector $\mathbf{X}_t$ is given by the prediction from the dynamics in (9). Using the expected value as the prediction, and by abusing the notation, we say that the state $\mathbf{X}_t$ is given as the solution of the first-order system of differential equations

$$d\mathbf{X}_t = A\mathbf{X}_t\, dt \tag{11}$$

By substituting iteratively into the discrete-time dynamics, one obtains that:
$p = 1, \mathbf{X}_t = X_{1(t)}$ and $dX_{1(t)} = -\alpha_1 X_{1(t)}dt$
$p = 2, dt = 1,\ X_{1(t+2)} \approx (2-\alpha_1)X_{1(t+1)} + (\alpha_1 - \alpha_2 - 1)X_{1(t)}$

$p = 3$,

$$\begin{aligned}
X_{1(t+1)} - X_{1(t)} &= X_{2(t)}dt \\
X_{2(t+1)} - X_{2(t)} &= X_{3(t)}dt \\
X_{3(t+1)} - X_{3(t)} &= -\alpha_3 X_{1(t)}dt - \alpha_2 X_{2(t)}dt - \alpha_1 X_{3(t)}dt \\
X_{1(t+2)} - X_{1(t+1)} &= X_{2(t+1)}dt \\
X_{2(t+2)} - X_{2(t+1)} &= X_{3(t+1)}dt \\
X_{3(t+2)} - X_{3(t+1)} &= -\alpha_3 X_{1(t+1)}dt - \alpha_2 X_{2(t+1)}dt - \alpha_1 X_{3(t+1)}dt \\
X_{1(t+3)} - X_{1(t+2)} &= X_{2(t+2)}dt \\
X_{2(t+3)} - X_{2(t+2)} &= X_{3(t+2)}dt \\
X_{3(t+3)} - X_{3(t+2)} &= -\alpha_3 X_{1(t+2)}dt - \alpha_2 X_{2(t+2)}dt - \alpha_1 X_{3(t+2)}dt
\end{aligned}$$

substituting into the X_1 dynamics and setting $dt = 1$:

$$\begin{aligned}
X_{1(t+3)} &\approx (3 - \alpha_1)X_{1(t+2)} + (2\alpha_1 - \alpha_2 - 3)X_{1(t+1)} \\
&+ (-\alpha_1 + \alpha_2 - \alpha_3 + 1)X_{1(t)}
\end{aligned} \tag{12}$$

Now, we approximate by Euler discretization to get the following for $X_{1(t)}$, $X_{2(t)}$ and $X_{3(t)}$. For $X_{3(t)}$ and using a time step of length 2 ($dt = 2$), we obtain

$$X_{3(t+2)} - X_{3(t)} = -\alpha_3 X_{1(t)} \cdot 2 - \alpha_2 X_{2(t)} \cdot 2 - \alpha_1 X_{3(t)}) \cdot 2\,.$$

Using the Euler approximation on $X_{2(t)}$ with time step 1 yields

$$X_{2(t+1)} - X_{2(t)} = X_{3(t)}$$

and similarly for X_{1t} we get

$$X_{1(t+1)} - X_{1(t)} = X_{2(t)}$$

and

$$X_{1(t+2)} - X_{1(t+1)} = X_{2(t+1)}$$

Hence, inserting in the approximation of $X_{3(t)}$ we find

$$X_{3(t+2)} = (1 - 2\alpha_1 + 2\alpha_2 - 2\alpha_3)X_{1(t)} + (4\alpha_1 - 2\alpha_2 - 2)X_{1(t+1)} + (1 - 2\alpha_1)X_{1(t+2)} \tag{13}$$

Thus, we see that we can recover the state of $X_{3(t)}$ by inserting $X_{1(t)} = T_t - \Lambda_t$ at times $t, t-1$ and $t-2$. Next, we have

$$X_{2(t+2)} - X_{2(t+1)} = X_{3(t+1)}$$

which implies, using the recursion on $X_{3(t+2)}$ in Equation (13)

$$X_{2(t+2)} = X_{2(t+1)} + (1 - 2\alpha_1 + 2\alpha_2 - 2\alpha_3)X_{1(t-1)} - (4\alpha_1 - 2\alpha_2 - 2)X_{1(t)} + (1 - 2\alpha_1)X_{1(t+1)}\,.$$

Inserting for $X_{2(t+1)}$, we get

$$X_{2(t+2)} = X_{1(t+2)} - 2\alpha_1 X_{1(t+1)} + (-2 + 4\alpha_1 - 2\alpha_2)X_{1(t)} + (1 - 2\alpha_1 + 2\alpha_2 - 2\alpha_3)X_{1(t-1)} \tag{14}$$

We see that $X_{2(t+2)}$ can be recovered by the temperature observation at times $t+2$, $t+1$, t and $t-1$. Finally, the state of $X_{1(t)}$ is obviously simply today's temperature less its seasonal state.

4 Temperature futures pricing

As temperature is not a tradable asset in the market place, no replication arguments hold for any temperature futures and incompleteness of the market follows. In this context all equivalent measures Q will be risk-neutral probabilities. We assume the existence of a pricing measure Q, which can be parametrized and complete the market, Karatzas and Shreve (2001). For that, we pin down an equivalent measure $Q = Q_{\theta_t}$ to compute the arbitrage free price of a temperature future:

$$F_{(t,\tau_1,\tau_2)} = \mathrm{E}^{Q_{\theta_t}}\left[Y(T_t)|\mathcal{F}_t\right] \tag{15}$$

with $Y(T_t)$ being the payoff from the temperature index (CAT, HDD, CDD indices) and θ_t denotes the time dependent market price of risk (MPR). The risk adjusted probability measure can be retrieved via Girsanov's theorem, by establishing:

$$B_t^\theta = B_t - \int_0^t \theta_u du \tag{16}$$

B_t^θ is a Brownian motion for any time before the end of the trading time ($t \leq \tau_{\max}$) and a martingale under Q_{θ_t}. Here the market price of risk (MPR) $\theta_t = \theta$ is as a real valued, bounded and piecewise continuous function. Under Q_θ, the temperature dynamics of (10) become

$$d\mathbf{X}_t = (A\mathbf{X}_t + \mathbf{e}_p\sigma_t\theta_t)dt + \mathbf{e}_p\sigma_t dB_t^\theta \tag{17}$$

with explicit dynamics, for $s \geq t \geq 0$:

$$\begin{aligned} \mathbf{X}_s &= \exp\{A(s-t)\}\mathbf{x} + \int_t^s \exp\{A(s-u)\}\mathbf{e}_p\sigma_u\theta_u du \\ &+ \int_t^s \exp\{A(s-u)\}\mathbf{e}_p\sigma_u dB_u^\theta \end{aligned} \tag{18}$$

From Theorem 4.2 (page 12) in Karatzas and Shreve (2001) we can parametrize the market price of risk θ_t and relate it to the risk premium for traded assets (as WD are indeed tradable assets) by the equation

$$\mu_t + \delta_t - r_t = \sigma_t\theta_t \tag{19}$$

where μ_t is the mean rate of return process, δ_t defines a dividend rate process, σ_t denotes the volatility process and r_t determines the risk-free interest rate process of the traded asset. In other words, the risk premium is the compensation, in terms of mean growth rate, for taking additional risk when investing in the traded asset. Assuming that $\delta_t = 0$, a sufficient parametrization of the MPR is setting $\theta_t = (\mu_t - r_t)/\sigma_t$ to make the discounted asset prices martingales. We later relax that assumption, by considering the time dependent market price of risk.

4.1 CAT Futures and Options

Following Equation (15) and using Fubini-Tonelli, the risk neutral price of a future based on a CAT index under Q_θ is defined as:

$$F_{CAT(t,\tau_1,\tau_2)} = \mathrm{E}^{Q_\theta}\left[\int_{\tau_1}^{\tau_2} T_s ds|\mathcal{F}_t\right] \tag{20}$$

For contracts whose trading date is earlier than the temperature measurement period, i.e. $0 \leq t \leq \tau_1 < \tau_2$, Benth et al. (2007) calculate the future price explicitly by inserting the temperature model (7) into (20):

$$
\begin{aligned}
F_{CAT(t,\tau_1,\tau_2)} &= \int_{\tau_1}^{\tau_2} \Lambda_u du + \mathbf{a}_{t,\tau_1,\tau_2} \mathbf{X}_t + \int_t^{\tau_1} \theta_u \sigma_u \mathbf{a}_{t,\tau_1,\tau_2} \mathbf{e}_p du \\
&+ \int_{\tau_1}^{\tau_2} \theta_u \sigma_u \mathbf{e}_1^\top A^{-1} \left[\exp\left\{ A(\tau_2 - u) \right\} - I_p \right] \mathbf{e}_p du
\end{aligned} \tag{21}
$$

with $\mathbf{a}_{t,\tau_1,\tau_2} = \mathbf{e}_1^\top A^{-1} \left[\exp\left\{ A(\tau_2 - t) \right\} - \exp\left\{ A(\tau_1 - t) \right\} \right]$ and $p \times p$ identity matrix I_p. While for CAT futures traded between the measurement period i.e. $\tau_1 \leq t < \tau_2$, the risk neutral price is:

$$
\begin{aligned}
F_{CAT}(t, \tau_1, \tau_2) &= \mathrm{E}^{Q_\theta} \left[\int_{\tau_1}^{t} T_s ds | \mathcal{F}_t \right] + \mathrm{E}^{Q_\theta} \left[\int_{t}^{\tau_2} T_s ds | \mathcal{F}_t \right] \\
&= \mathrm{E}^{Q_\theta} \left[\int_{\tau_1}^{t} T_s ds | \mathcal{F}_t \right] + \int_t^{\tau_2} \Lambda_u du + \mathbf{a}_{t,t,\tau_2} \mathbf{X}_t \\
&+ \int_t^{\tau_2} \theta_u \sigma_u \mathbf{e}_1^\top A^{-1} \left[\exp\left\{ A(\tau_2 - u) \right\} - I_p \right] \mathbf{e}_p du
\end{aligned}
$$

where $\mathbf{a}_{t,t,\tau_2} = \mathbf{e}_1^\top A^{-1} \left[\exp\left\{ A(\tau_2 - t) \right\} - I_p \right]$. Since the expected value of the temperature from τ_1 to t is already known, this time the future price consists on a random and a deterministic part. Details of the proof can be found in Benth, Saltyte Benth and Koekebakker (2008). Note that the CAT futures price is given by the aggregated mean temperature (seasonality) over the measurement period plus a mean reversion weighted dependency on X_t, which is depending on the temperature of previous days $T_{t-k}, k \leq p$. The last two terms smooth the market price of risk over the period from the trading date t to the end of the measurement period τ_2, with a change happening in time τ_1. Similar results hold for the C24AT index futures.

Note that that only coordinate of $\mathbf{X}_t$ that has a has a random component dB^θ is X_{pt}, hence the dynamics under Q_θ of $F_{CAT}(t, \tau_1, \tau_2)$ is:

$$
dF_{CAT}(t, \tau_1, \tau_2) = \sigma_t \mathbf{a}_{t,\tau_1,\tau_2} \mathbf{e}_p dB_t^\theta
$$

where $\sigma_t \mathbf{a}_{t,\tau_1,\tau_2} \mathbf{e}_p$ denotes CAT future volatility.

From the risk neutral dynamics of $F_{CAT}(t, \tau_1, \tau_2)$, the explicit formulae for the CAT call option written on a CAT future with strike K at exercise time $\tau < \tau_1$ during the period $[\tau_1, \tau_2]$:

$$
\begin{aligned}
C_{CAT(t,\tau,\tau_1,\tau_2)} &= \exp\left\{ -r(\tau - t) \right\} \\
&\times \Big[\left(F_{CAT(t,\tau_1,\tau_2)} - K \right) \Phi \left\{ d\left(t, \tau, \tau_1, \tau_2\right) \right\} \\
&+ \int_t^{\tau} \Sigma^2_{CAT(s,\tau_1,\tau_2)} ds \Phi \left\{ d\left(t, \tau, \tau_1, \tau_2\right) \right\} \Big]
\end{aligned} \tag{22}
$$

where

$$
d\left(t, \tau, \tau_1, \tau_2\right) = F_{CAT(t,\tau_1,\tau_2)} - K \Big/ \sqrt{\int_t^{\tau} \Sigma^2_{CAT(s,\tau_1,\tau_2)} ds}
$$

and

$$
\Sigma_{CAT(s,\tau_1,\tau_2)} = \sigma_t \mathbf{a}_{t,\tau_1,\tau_2} \mathbf{e}_p
$$

Note that once that a risk neutral probability Q_θ is chosen, the market of futures and options is complete and therefore we can replicate the option. In order to do that, one should compute the number of CAT-futures held in the portfolio, which is simply computed by the option's delta:

$$\frac{\partial C_{CAT(t,\tau,\tau_1,\tau_2)}}{\partial F_{CAT(t,\tau_1,\tau_2)}} = \Phi\left\{d\left(t,T,\tau_1,\tau_2\right)\right\} \tag{23}$$

The strategy holds close to zero CAT futures when the option is far out of the money, close to 1 otherwise.

4.2 CDD Futures and Options

Analogously, one derives the CDD future price. Following (15), the risk neutral price of a CDD future which is traded at $0 \leq t \leq \tau_1 < \tau_2$ is defined as:

$$\begin{aligned} F_{CDD(t,\tau_1,\tau_2)} &= \mathrm{E}^{Q_\theta}\left[\int_{\tau_1}^{\tau_2} \max(T_s - c, 0) ds | \mathcal{F}_t\right] \\ &= \int_{\tau_1}^{\tau_2} \upsilon_{t,s} \psi\left[\frac{m_{\{t,s,\mathbf{e}_1^\top \exp\{A(s-t)\}\mathbf{X}_t\}} - c}{\upsilon_{t,s}}\right] ds \end{aligned} \tag{24}$$

where

$$m_{\{t,s,x\}} = \Lambda_s - c + \int_t^s \sigma_u \theta_u \mathbf{e}_1^\top \exp\left\{A(s-t)\right\} \mathbf{e}_p du + x$$

$$\upsilon_{t,s}^2 = \int_t^s \sigma_u^2 \left[\mathbf{e}_1^\top \exp\left\{A(s-t)\right\} \mathbf{e}_p\right]^2 du$$

$$\psi(x) = x\Phi(x) + \varphi(x) \tag{25}$$

For CDD futures contracts traded at $\tau_1 \leq t \leq \tau_2$, the non-abitrage price of a CDD future is:

$$\begin{aligned} F_{CDD(t,\tau_1,\tau_2)} &= \mathrm{E}^{Q_\theta}\left[\int_{\tau_1}^{\tau_2} \max(T_s - c, 0) ds | \mathcal{F}_t\right] \\ &= \mathrm{E}^{Q_\theta}\left[\int_{\tau_1}^{t} \max(T_s - c, 0) ds | \mathcal{F}_t\right] \\ &+ \int_{t}^{\tau_2} \upsilon_{t,s} \psi\left[\frac{m_{\{t,s,\mathbf{e}_1^\top \exp\{A(s-t)\}\mathbf{X}_t\}} - c}{\upsilon_{t,s}}\right] ds \end{aligned} \tag{26}$$

with $m_{\{t,s,x\}}$ and $\upsilon_{t,s}^2$ defined as above. Note again that the expected value of the temperature from τ_1 to t is known.

The dynamics of the $F_{CDD(t,\tau_1,\tau_2)}$ for $0 \leq t \leq \tau_1$ under Q_θ is given by:

$$\begin{aligned} dF_{CDD}(t,\tau_1,\tau_2) &= \sigma_t \int_{\tau_1}^{\tau_2} \mathbf{e}_1^\top \exp\left\{A(s-t)\right\} \mathbf{e}_p \\ &\times \Phi\left[\frac{m_{\{t,s,\mathbf{e}_1^\top \exp\{A(s-t)\}\mathbf{X}_t\}} - c}{\upsilon_{t,s}}\right] ds dB_t^\theta \end{aligned}$$

The term structre of volatility for CDD futures is defined as:

$$\begin{aligned} \Sigma_{CDD(s,\tau_1,\tau_2)} &= \sigma_t \int_{\tau_1}^{\tau_2} \mathbf{e}_1^\top \exp\left\{A(s-t)\right\} \mathbf{e}_p \\ &\times \Phi\left[\frac{m_{\{t,s,\mathbf{e}_1^\top \exp\{A(s-t)\}\mathbf{X}_t\}} - c}{\upsilon_{t,s}}\right] ds \end{aligned} \tag{27}$$

For the call option written CDD-future, the solution is not analytical but is given in terms of an expression suitable for Monte Carlo simulation. The risk neutral price of a CDD call written on a CDD future with strike K at exercise time $\tau < \tau_1$ during the period $[\tau_1, \tau_2]$:

$$\begin{aligned} C_{CDD(t,T,\tau_1,\tau_2)} &= \exp\{-r(\tau - t)\} \\ &\times \mathrm{E}\left[\max\left\{\int_{\tau_1}^{\tau_2} \upsilon_{\tau,s}\psi\left(\frac{m_{\text{index}} - c}{\upsilon_{\tau,s}}\right) ds - K, 0\right\}\right]_{\mathbf{x}=\mathbf{X}_t} \end{aligned} \tag{28}$$

$$\text{index} = \tau, s, \mathbf{e}_1^\top \exp\{A(s-t)\}\mathbf{x} + \int_t^\tau \mathbf{e}_1^\top \exp\{A(s-u)\}\mathbf{e}_p\sigma_u\theta_u du + \Sigma_{(s,t,\tau)}Y$$

$$Y \sim \mathrm{N}(0,1)$$

$$\Sigma^2_{(s,t,\tau)} = \int_t^\tau \left[\mathbf{e}_1^\top \exp\{A(s-u)\}\mathbf{e}_p\right]^2 \sigma_u^2 du$$

If the $\Sigma_{CDD(s,\tau_1,\tau_2)}$ is non-zero for almost everywhere $t \in [0, \tau]$, then the hedging strategy H_{CDD} is given by:

$$\begin{aligned} H_{CDD(t,\tau_1,\tau_2)} &= \frac{\sigma_t}{\Sigma_{CDD(s,\tau_1,\tau_2)}} \mathrm{E}\left[1\left\{\int_{\tau_1}^{\tau_2} \upsilon_{\tau,s}\psi\left(\frac{m_{(\tau,s,Z(x))} - c}{\upsilon_{\tau,s}}\right) ds > K\right\}\right. \\ &\times \left.\int_{\tau_1}^{\tau_2} \mathbf{e}_1^\top \exp\{A(s-t)\}\mathbf{e}_p\Phi\left(\frac{m_{(\tau,s,Z(x))} - c}{\upsilon_{\tau,s}}\right) ds\right]_{\mathbf{x}=\mathbf{X}_t} \end{aligned} \tag{29}$$

for $t \leq \tau$, where $Z(x)$ is a normal random variable with mean

$$\mathbf{e}_1^\top \exp\{A(s-t)\}\mathbf{x} + \int_t^\tau \mathbf{e}_1^\top \exp\{A(s-u)\}\mathbf{e}_p\sigma_u\theta_u du$$

and variance $\Sigma^2_{(s,t,\tau)}$.

4.3 Infering the market price of temperature risk

In the weather derivative market there is obviously the question of choosing the right price among possible arbitrage free prices. For pricing nontradable assets one essentially needs to incorporate the market price of risk (MPR), which is an important parameter of the associated equivalent martingale measures used to price and hedge weather futures/options in the market. MPR can be calibrated from data and thereby using the market to pin down the price of the temperature derivative. Once we know the MPR for temperature futures, then we know the MPR for options.

By inverting (21), given observed prices, θ_t is inferred for contracts with trading date $t \leq \tau_1 < \tau_2$. Setting θ_t^i as a constant for each of the i contract, with $i = 1 \ldots 7$, $\hat{\theta}_t^i$ is estimated via:

$$\begin{aligned} \arg\min_{\hat{\theta}_t^i} & \left(F_{AAT(t,\tau_1^i,\tau_2^i)} - \int_{\tau_1^i}^{\tau_2^i} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t,\tau_1^i,\tau_2^i}\widehat{\mathbf{X}}_t \right. \\ & - \hat{\theta}_t^i \left\{ \int_t^{\tau_1^i} \hat{\sigma}_u \hat{\mathbf{a}}_{t,\tau_1^i,\tau_2^i}\mathbf{e}_p du \right. \\ & \left.\left. + \int_{\tau_1^i}^{\tau_2^i} \hat{\sigma}_u \mathbf{e}_1^\top A^{-1}\left[\exp\{A(\tau_2^i - u)\} - I_p\right]\mathbf{e}_p du \right\} \right)^2 \end{aligned} \tag{30}$$

A simpler parametrization of θ_t is to assume that it is constant for all maturities. We therefore estimate this constant θ_t for all contracts with $t \leq \tau_1^i < \tau_2^i$, $i = 1, \cdots, 7$ as follows:

$$
\begin{aligned}
\arg\min_{\hat{\theta}_t} \Sigma_{i=1}^{7} \quad \Bigg(\quad & F_{CAT(t,\tau_1^i,\tau_2^i)} - \int_{\tau_1^i}^{\tau_2^i} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t,\tau_1^i,\tau_2^i} \mathbf{X}_t \\
- \quad & \hat{\theta}_t \left\{ \int_t^{\tau_1^i} \hat{\sigma}_u \hat{\mathbf{a}}_{t,\tau_1^i,\tau_2^i} \mathbf{e}_p du \right. \\
+ \quad & \left. \int_{\tau_1^i}^{\tau_2^i} \hat{\sigma}_u \mathbf{e}_1^\top A^{-1} \left[\exp\left\{A(\tau_2^i - u)\right\} - I_p \right] \mathbf{e}_p du \right\} \Bigg)^2
\end{aligned}
$$

Assuming now that, instead of one constant market price of risk per trading day, we have a step function with jump $\hat{\theta}_t = I\left(u \leq \xi\right) \hat{\theta}_t^1 + I\left(u > \xi\right) \hat{\theta}_t^2$ with jump point ξ (take e.g. the first 150 days before the beginning of the measurement period). Then we estimate $\hat{\theta}_t$ for contracts with $t \leq \tau_1^i < \tau_2^i$, $i = 1, \cdots, 7$ by:

$$
\begin{aligned}
f(\xi) \quad = \quad & \arg\min_{\hat{\theta}_t^1, \hat{\theta}_t^2} \Sigma_{i=1}^{7} \Bigg(F_{CAT(t,\tau_1^i,\tau_2^i)} - \int_{\tau_1^i}^{\tau_2^i} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t,\tau_1^i,\tau_2^i} \mathbf{X}_t \\
- \quad & \hat{\theta}_t^1 \left\{ \int_t^{\tau_1^i} I\left(u \leq \xi\right) \hat{\sigma}_u \hat{\mathbf{a}}_{t,\tau_1^i,\tau_2^i} \mathbf{e}_p du \right. \\
+ \quad & \left. \int_{\tau_1^i}^{\tau_2^i} I\left(u \leq \xi\right) \hat{\sigma}_u \mathbf{e}_1^\top A^{-1} \left[\exp\left\{A(\tau_2^i - u)\right\} - I_p \right] \mathbf{e}_p du \right\} \\
- \quad & \hat{\theta}_t^2 \left\{ \int_t^{\tau_1^i} I\left(u > \xi\right) \hat{\sigma}_u \hat{\mathbf{a}}_{t,\tau_1^i,\tau_2^i} \mathbf{e}_p du \right. \\
+ \quad & \left. \int_{\tau_1^i}^{\tau_2^i} I\left(u > \xi\right) \hat{\sigma}_u \mathbf{e}_1^\top A^{-1} \left[\exp\left\{A(\tau_2^i - u)\right\} - I_p \right] \mathbf{e}_p du \right\} \Bigg)^2
\end{aligned}
$$

Generalising the previous piecewise continuous function, the (inverse) problem of determining θ_t with $t \leq \tau_1^i < \tau_2^i$, $i = 1, \cdots 7$ can be formulated via a series expansion for θ_t:

$$
\begin{aligned}
\arg\min_{\hat{\gamma}_k} \Sigma_{i=1}^{7} \quad \Bigg(\quad & F_{AAT(t,\tau_1^i,\tau_2^i)} - \int_{\tau_1^i}^{\tau_2^i} \hat{\Lambda}_u du - \hat{\mathbf{a}}_{t,\tau_1{}^i,\tau_2^i} \widehat{\mathbf{X}}_t \\
- \quad & \int_t^{\tau_1^i} \sum_{k=1}^{K} \hat{\gamma}_k \hat{h}_k(u_i) \hat{\sigma}_{u_i} \hat{\mathbf{a}}_{t,\tau_1,\tau_2} \mathbf{e}_p du_i \\
- \quad & \int_{\tau_1^i}^{\tau_2^i} \sum_{k=1}^{K} \hat{\gamma}_k \hat{h}_k(u_i) \hat{\sigma}_{u_i} \mathbf{e}_1^\top A^{-1} \left[\exp\left\{A(\tau_2^i - u_i)\right\} \right. \\
- \quad & \left. I_p \right] \mathbf{e}_p du_i \Bigg)^2
\end{aligned} \tag{31}
$$

where $h_k(u_i)$ is a vector of known basis functions and γ_k defines the coefficients. Here $h_k(u_i)$ may denote a spline basis for example. Härdle and López Cabrera (2009) show additional methods about how to infere the MPR.

City	Period	$\hat{a}_0$	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_3$
Tokyo	19730101-20081231	15.76	7.82e-05	10.35	-149.53
Osaka	19730101-20081231	15.54	1.28e-04	11.50	-150.54
Beijing	19730101-20081231	11.97	1.18e-04	14.91	-165.51
Taipei	19920101-20090806	23.21	1.68e-03	6.78	-154.02

Table 2: Seasonality estimates of daily average temperatures in Asia. Data source: Bloomberg

City	$\hat{\tau}$(p-value)	$\hat{k}$(p-value)
Tokyo	-56.29(< 0.01)	0.091(< 0.1)
Osaka	-17.86(< 0.01)	0.138(< 0.1)
Beijing	-20.40(< 0.01)	0.094(< 0.1)
Taipei	-33.21(< 0.01)	0.067(< 0.1)

Table 3: Stationarity tests.

5 Asian temperature derivatives

5.1 Asian temperature dynamics

We turn now to the analysis of the weather dynamics for Tokyo, Osaka, Beijing and Taipei daily temperature data. The temperature data were obtained from the Tokyo Narita International Airport, Osaka Kansai International Airport and Bloomberg. We consider recordings of daily average temperatures from 19730101 - 20090604. In all studied data, a linear trend was not detectable but a clear seasonal pattern emerged. Figure 1 shows 8 years of daily average temperatures and the least squares fitted seasonal function with trend:

$$\Lambda_t = a_0 + a_1 t + a_2 cos \left\{ \frac{2\pi(t - a_3)}{365} \right\} \tag{32}$$

The estimated coefficients are displayed in Table 2.

The low order polynomial deterministic trend smooths the seasonal pattern and makes the model to be parsimonius. The coefficient $\hat{a}_0$ can be interpretated as the average temperature, while $\hat{a}_1$ as the global warming trend component. In most of the Asian cases, as expected, the low temperatures are observed in the winter and high temperatures in the summer.

After removing the seasonality in (32) from the daily average temperatures,

$$X_t = T_t - \Lambda_t \tag{33}$$

we check whether X_t is a stationary process $I(0)$. In order to do that, we apply the Augmented Dickey-Fuller test (ADF) $(1-L)X = c_1 + \mu t + \tau LX + \alpha_1(1-L)LX + \dots \alpha_p(1-L)L^p X + \varepsilon_t$, where p is the number of lags by which the regression is augmented to get residuals free of autocorrelation. Under H_0 (unit root), τ should be zero. Therefore the test statistic of the OLS estimator of τ is applicable. If the null hypothesis H_0 ($\tau = 0$) is rejected then X_t is a stationary process $I(0)$.

Stationarity can also be verified by using the KPSS Test: $X_t = c + \mu t + k \sum_{i=1}^{t} \xi_i + \varepsilon_t$ with stationary ε_t and iid ξ_t with an expected value 0 and variance 1. If $H_0 : k = 0$ is accepted then the process is stationary. The estimates of τ and k of the previuos stationarity tests are illustrated in Table 3, indicating that the stationarity is achieved.

The Partial Autocorrelation Function (PACF) of (33) suggests that higher order autoregressive models $AR(p), p > 1$ are suitable for modelling the time evolution of Asia temperatures after removing seasonality, see Figure 2.

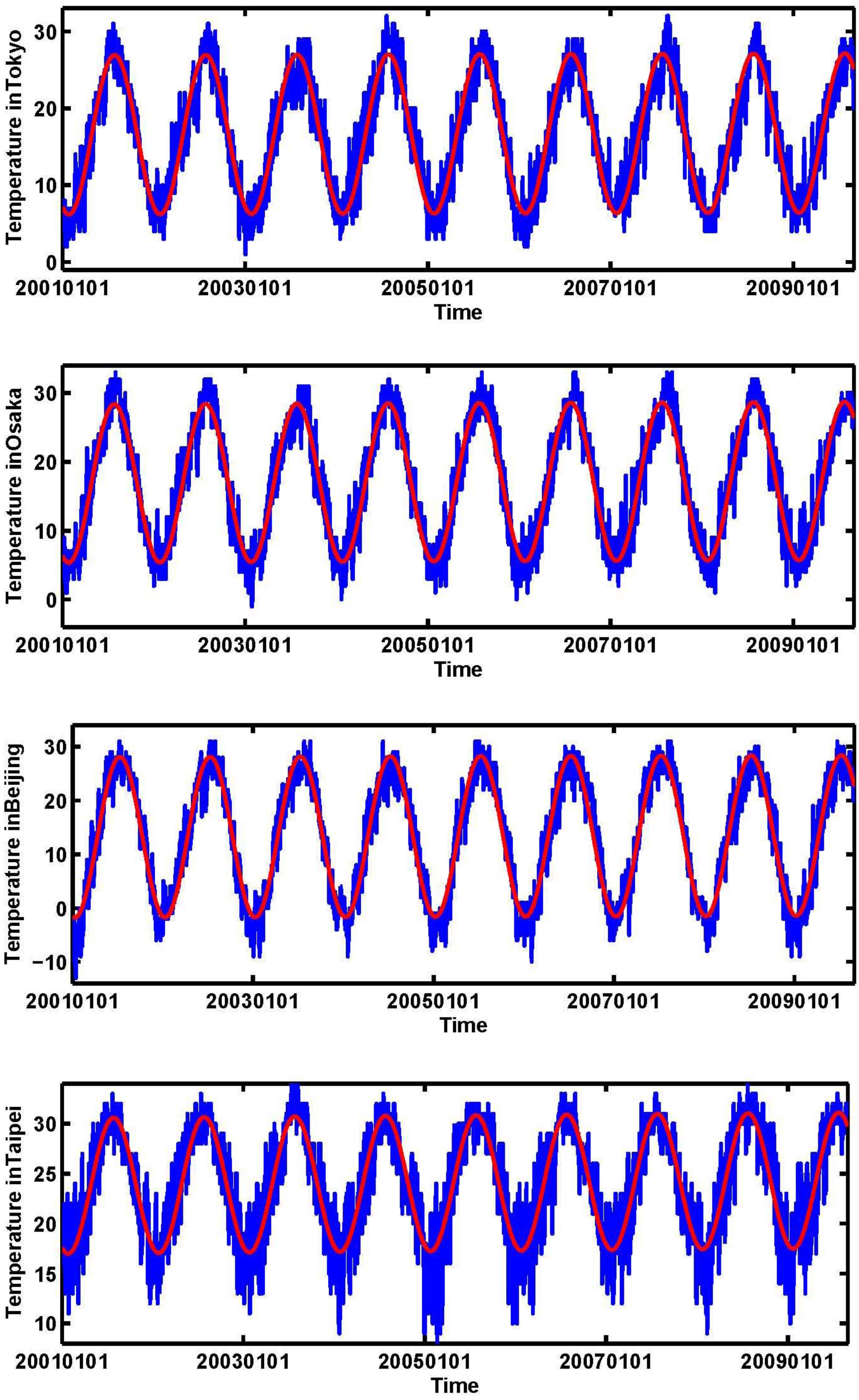

Figure 1: Seasonality effect and daily average temperatures for Tokyo Narita International Airport, Osaka Kansai International Airport, Beijing and Taipei.

AsianWeather1

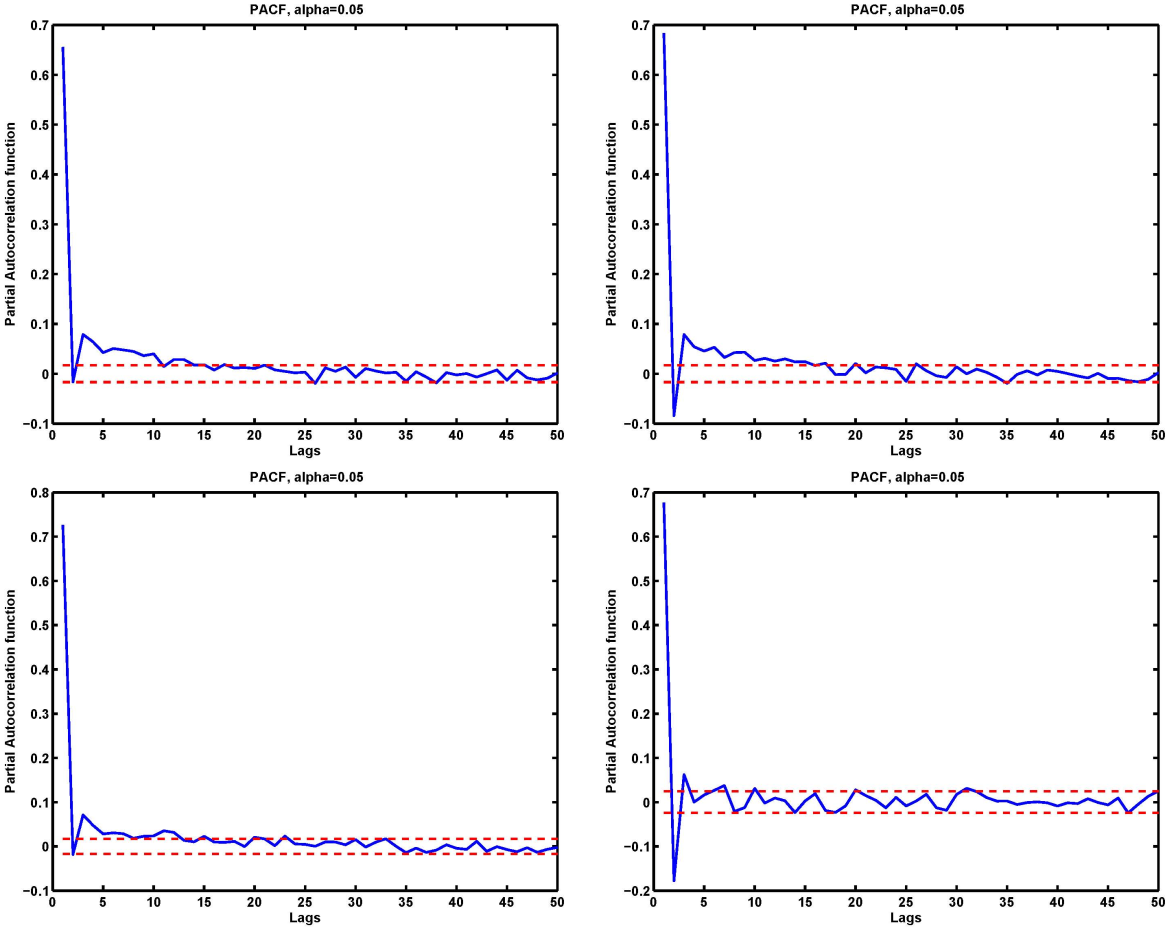

Figure 2: Partial autocorrelation function (PACF) for Tokyo (upper left), Osaka (upper right), Beijing (lower left), Taipei (lower right)

AsianWeather2

Year	every 3 years	every 6 years	every 9 years	every 12 years	every 18 years
73-75	AR(1)	AR(3)	AR(3)	AR(8)*	AR(9)*
76-78	AR(1)				
79-81	AR(1)	AR(8)*			
82-84	AR(8)*		AR(9)*		
85-87	AR(1)	AR(3)		AR(3)	
88-90	AR(1)				
91-93	AR(1)	AR(3)	AR(3)		AR(3)
94-96	AR(1)				
97-99	AR(1)	AR(1)		AR(3)	
00-02	AR(1)		AR(3)		
03-05	AR(3)	AR(3)			
06-09	AR(1)				

Table 4: Tokyo Moving window for AR, * denotes instability

<table>
<tr><th>Year</th><th>every 3 years</th><th>every 6 years</th><th>every 9 years</th><th>every 12 years</th><th>every 18 years</th></tr>
<tr><td>73-75</td><td>AR(1)</td><td rowspan="2">AR(3)</td><td rowspan="3">AR(3)</td><td rowspan="4">AR(3)</td><td rowspan="6">AR(6)*</td></tr>
<tr><td>76-78</td><td>AR(3)</td></tr>
<tr><td>79-81</td><td>AR(3)</td><td rowspan="2">AR(3)</td></tr>
<tr><td>82-84</td><td>AR(2)</td><td rowspan="3">AR(3)</td></tr>
<tr><td>85-87</td><td>AR(3)</td><td rowspan="2">AR(3)</td><td rowspan="4">AR(6)*</td></tr>
<tr><td>88-90</td><td>AR(3)</td></tr>
<tr><td>91-93</td><td>AR(3)</td><td rowspan="2">AR(3)</td><td rowspan="3">AR(6)*</td><td rowspan="6">AR(7)*</td></tr>
<tr><td>94-96</td><td>AR(1)</td></tr>
<tr><td>97-99</td><td>AR(2)</td><td rowspan="2">AR(2)</td><td rowspan="4">AR(7)*</td></tr>
<tr><td>00-02</td><td>AR(1)</td><td rowspan="3">AR(3)</td></tr>
<tr><td>03-05</td><td>AR(3)</td><td rowspan="2">AR(3)</td></tr>
<tr><td>06-09</td><td>AR(1)</td></tr>
</table>

Table 5: Osaka Moving window for AR, * denotes instability.

	Coefficient	Tokyo(p=3)	Osaka(p=3)	Beijing(p=3)	Taipei(p=3)
AR	β_1	0.668	0.748	0.741	0.808
	β_2	-0.069	-0.143	-0.071	-0.228
	β_3	-0.079	-0.079	0.071	0.063
CAR	α_1	-2.332	-2.252	-2.259	-2.192
	α_2	1.733	-1.647	-1.589	-1.612
	α_3	-0.480	-0.474	-0.259	-0.357
Eigenvalues	real part of λ_1	-1.257	-1.221	-0.231	-0.396
	real part of $\lambda_{2,3}$	-0.537	-0.515	-1.013	-0.8976

Table 6: Coefficients of (C)AR(p), Model selection: AIC.

The covariance stationarity dynamics were captured using autoregressive lags over different year-lengths moving windows, as it is denoted in Table 4 and Table 5 for the case of Tokyo and Osaka. The autoregressive models showed, for larger length periods, higher order p and sometimes lack of stability (AR *), i.e. the eigenvalues of matrix A (8) had positive real part. Since local estimates of the a fitted seasonal variation σ_t with GARCH models captures long memory affects and assuming that it shocks temperature residuals in the same way over different length periods, the autoregressive model AR(3) was therefore chosen. $p = 3$ is also confirmed by the Akaike and Schwarz information criteria for each city. The coefficients of the fitted autoregressive process

$$X_{t+p} = \sum_{i=1}^{p} \beta_i X_{t+p-i} + \sigma_t \varepsilon_t \tag{34}$$

and their corresponding are $CAR(3)$-parameters displayed in Table 6. The stationarity condition is fulfilled since the eigenvalues of A have negative real parts. The element components of the matrix A (8) do not change over time, this makes the process stable.

After trend and seasonal components were removed, the residuals ε_t and the squared residuals ε_t^2 of (34) for Chinese temperature data are plotted in the Figure 4 and for Japan in Figure 3. According to the modified Li-McLeod Portmanteau test, we reject at 1% significance level the null hypothesis H_0 that the residuals are uncorrelated. The ACF of the residuals of AR(3) for Asian cities is close to zero and according to Box-Ljung statistic the first few lags are insignificant for the case of China and Japan. However, the ACF for the squared residuals (also displayed in Figure 5) shows a high seasonal pattern.

This seasonal dependence of variance of residuals of the AR(3) ($\hat{\sigma}^2_{t,FTSG}$) for the Asian cities is calibrated with a truncanted Fourier function and a Generalized Autoregressive Conditional

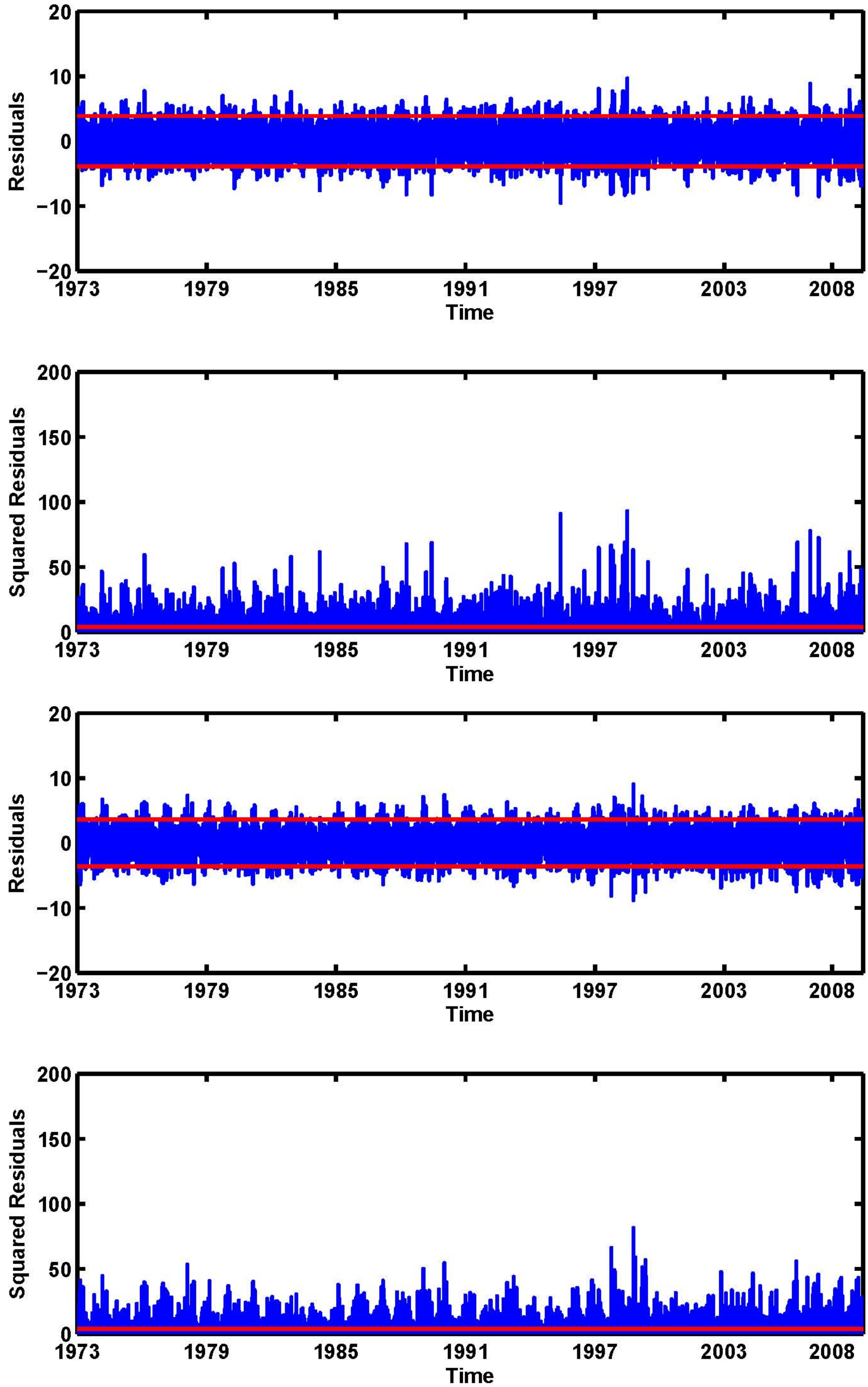

Figure 3: Residuals $\hat{\varepsilon}_t$ and squared residuals $\hat{\varepsilon}_t^2$ of the AR(p) (for Tokyo (1-2 panel) and Osaka (3-4 pannel)) during 19730101-20081231. No rejection of H_0 that the residuals are uncorrelated at 0% significance level, according to the modified Li-McLeod Portmanteau test

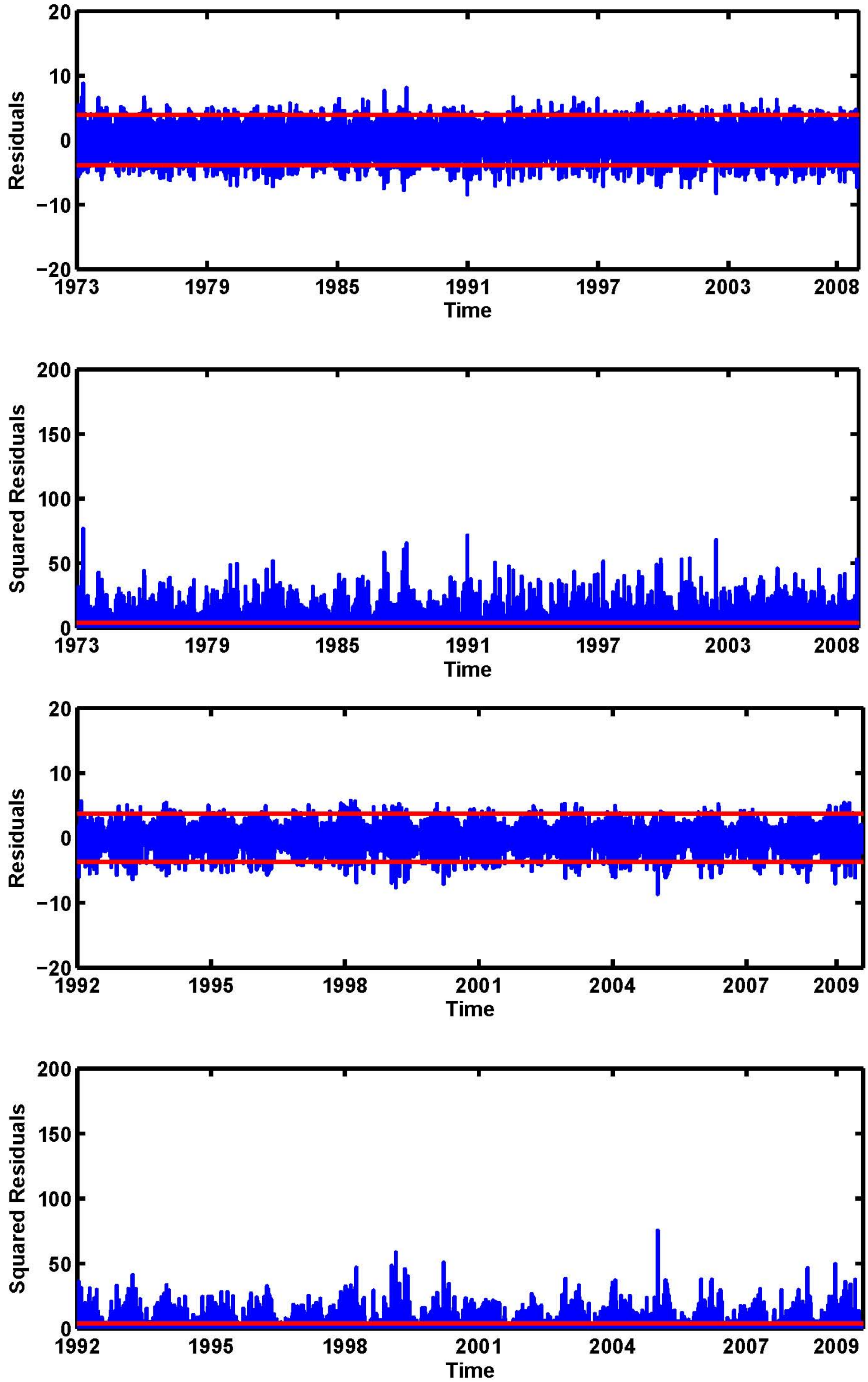

Figure 4: Residuals $\hat{\varepsilon}_t$ and squared residuals $\hat{\varepsilon}_t^2$ of the AR(p) (for Beijing (1-2 panel) and Taipei (3-4 pannel)) during 19730101-20081231. No rejection of H_0 that the residuals are uncorrelated at 0% significance level, according to the modified Li-McLeod Portmanteau test

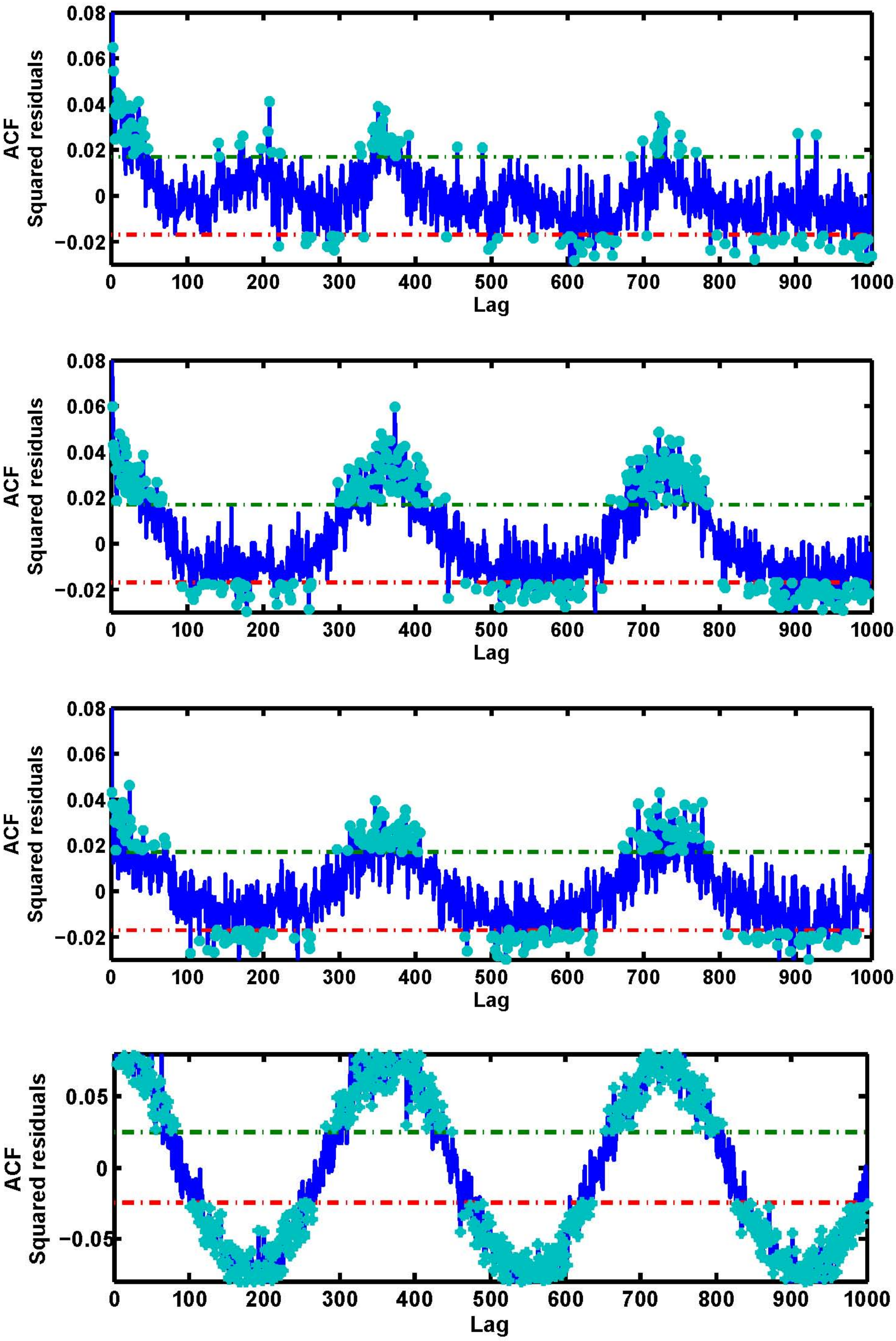

Figure 5: ACF for squared residuals $\hat{\varepsilon}_t^2$ of the AR(p) (for Tokyo (1 panel), Osaka (2 panel), Beijing (3 panel) and Taipei (4 pannel)) during 19730101-20081231

AsianWeather4

	$\hat{c}_1$	$\hat{c}_2$	$\hat{c}_3$	$\hat{c}_4$	$\hat{c}_5$	$\hat{c}_6$	$\hat{c}_7$	$\hat{c}_8$	$\hat{c}_9$	α	β
Tokyo	3.91	-0.08	0.74	-0.70	-0.37	-0.13	-0.14	0.28	-0.15	0.09	0.50
Osaka	3.40	0.76	0.81	-0.58	-0.29	-0.17	-0.07	0.01	-0.04	0.04	0.52
Beijing	3.95	0.70	0.82	-0.26	-0.50	-0.20	-0.17	-0.05	0.10	0.03	0.33
Taipei	3.54	1.49	1.62	-0.41	-0.19	0.03	-0.18	-0.11	-0.16	0.06	0.33

Table 7: First 9 Coefficients of σ_t^2 and $GARCH(p=1, q=1)$.

Heteroscedasticity GARCH(p,q):

$$\begin{aligned}
\hat{\sigma}^2_{t,FTSG} &= c_1 + \sum_{i=1}^{16} \left\{ c_{2i} \cos\left(\frac{2i\pi t}{365}\right) + c_{2i+1} \sin\left(\frac{2i\pi t}{365}\right) \right\} \\
&+ \alpha_1(\sigma^2_{t-1}\varepsilon_{t-1})^2 + \beta_1 \sigma^2_{t-1}
\end{aligned} \tag{35}$$

Alternatively to the seasonal variation of the 2 step model, one can smooth the data with a Local Linear Regression (LLN) $\hat{\sigma}^2_{t,LLR}$ estimator:

$$\min_{a,b} \sum_{i=1}^{365} \left(\hat{\sigma}^2_{t,LLR,i} - a(t) - b(t)(T_i - t) \right)^2 K\left(\frac{T_i - t}{h} \right) \tag{36}$$

Asympotically they can be approximated by Fourier series estimators. Table 7 shows the first 9 coefficients of the seasonal variation using the 2 steps model. Figure 6 shows the daily empirical variance (the average of 35 years squared residuals for each day of the year) and the fitted squared volatility function for the residuals $\hat{\sigma}^2_{t,FTSG}$ and $\hat{\sigma}^2_{t,LLR}$ using Epanechnikov Kernel und bandwidth bandwidth $h = 4.49, 4.49$ for the Chinese cities and $h = 3.79$ for Japanese cities at 10% significance level. The results are different to the Campbell and Diebold (2005) effect for American and European temperature data, high variance in earlier winter - spring and low variance in late winter - late summer.

Figure 7 shows the ACF plot of the Asian temperature squared residuals $\hat{\varepsilon}^2_t$, after dividing out the seasonal volatility $\hat{\sigma}^2_{t,LLR}$ from the regression residuals. The ACF plot of the residuals remain unchanged and now the ACF plot for squared residuals presents a non-seasonal pattern. Table 8 shows the statistics for the standardized residuals under different seasonal variations ($\frac{\hat{\varepsilon}_t}{\sigma_{t,FTS}}, \frac{\hat{\varepsilon}_t}{\sigma_{t,FTSG}}$ and $\frac{\hat{\varepsilon}_t}{\sigma_{t,LLR}}$). The estimator of the seasonal variation with local linear regression was the closer to the normal residuals. The acceptance of the null hyptohesis H_0 of normality is at 1% significance level.

The log Kernel smoothing density estimate against a log Normal Kernel evaluated at 100 equally spaced points for Asian temperature residuals has been plotted in Figure (8) to verify if residuals become normally distributed. The seasonal variation modelled with a GARCH (1,1) and by the local linear regression are adequately capturing the intertemporal dependencies in daily temperature.

5.2 Pricing Asian futures

In this section, using Equation (30) and (31) but for C24AT index futures, we infered the market price of risk for C24AT Asian temperature derivatives as Härdle and López Cabrera (2009) did for Berlin monthly CAT futures. Table 9 shows the replication of the observed Tokyo C24AT index futures prices traded in Bloomberg on 20090130, using the constant MPR for each contract per trading day and the time dependent MPR using cubic polynomials with number of knots equal

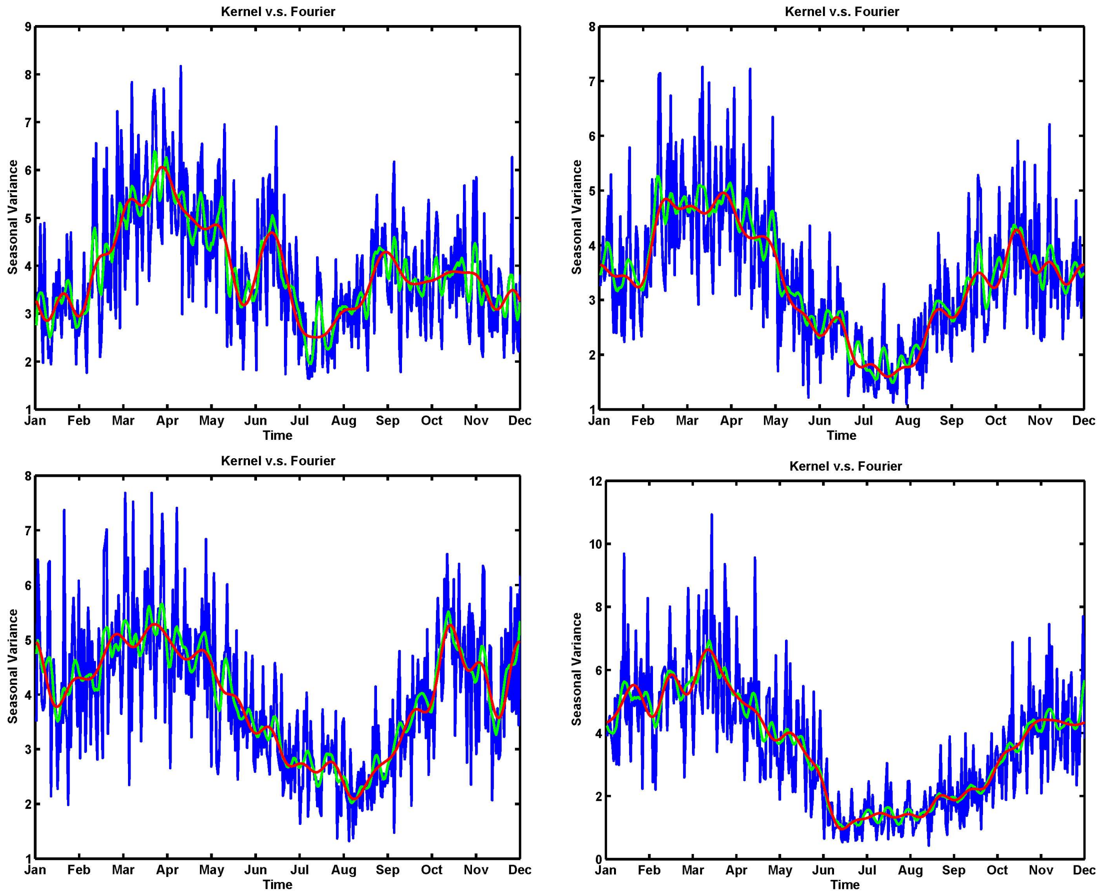

Figure 6: Daily empirical variance, $\hat{\sigma}^2_{t,FTSG}$, $\hat{\sigma}^2_{t,LLR}$ for Tokyo (upper left), Osaka (upper right), Beijing (lower left), Tapei (lower right)

AsianWeather5

City		$\frac{\hat{\varepsilon}_t}{\sigma_{t,FTS}}$	$\frac{\hat{\varepsilon}_t}{\sigma_{t,FTSG}}$	$\frac{\hat{\varepsilon}_t}{\sigma_{t,LLR}}$
Tokyo	Jarque Bera	158.00	127.23	114.50
	Kurtosis	3.46	3.39	3.40
	Skewness	-0.15	-0.11	-0.12
Osaka	Jarque Bera	129.12	119.71	105.02
	Kurtosis	3.39	3.35	3.33
	Skewness	-0.15	-0.14	-0.14
Beijing	Jarque Bera	234.07	223.67	226.09
	Kurtosis	3.28	3.27	3.25
	Skewness	-0.29	-0.29	-0.29
Taipei	Jarque Bera	201.09	198.40	184.17
	Kurtosis	3.36	3.32	3.3
	Skewness	-0.39	-0.39	-0.39

Table 8: Statistics of the Asian temperature residuals $\hat{\varepsilon}_t$ and squared residuals $\hat{\varepsilon}_t^2$, after dividing out the seasonal volatility $\hat{\sigma}^2_{t,LLR}$ from the regression residuals

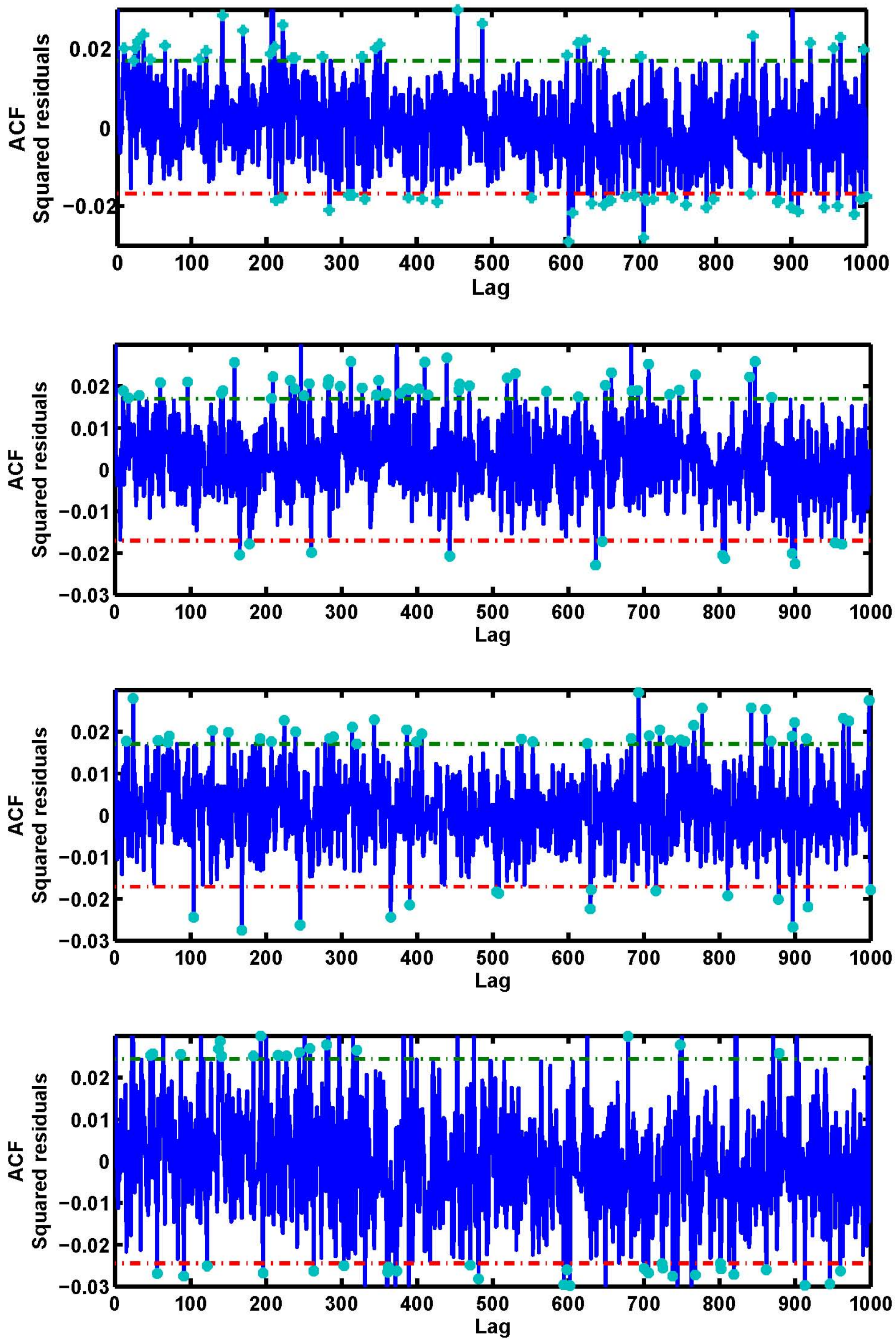

Figure 7: ACF for temperature (squared) residuals $\frac{\hat{\varepsilon}_t}{\sigma_{t,LLR}}$ for Tokyo (1 panel), Osaka (2 panel), Beijing (3 panel) and Taipei (4 panel).

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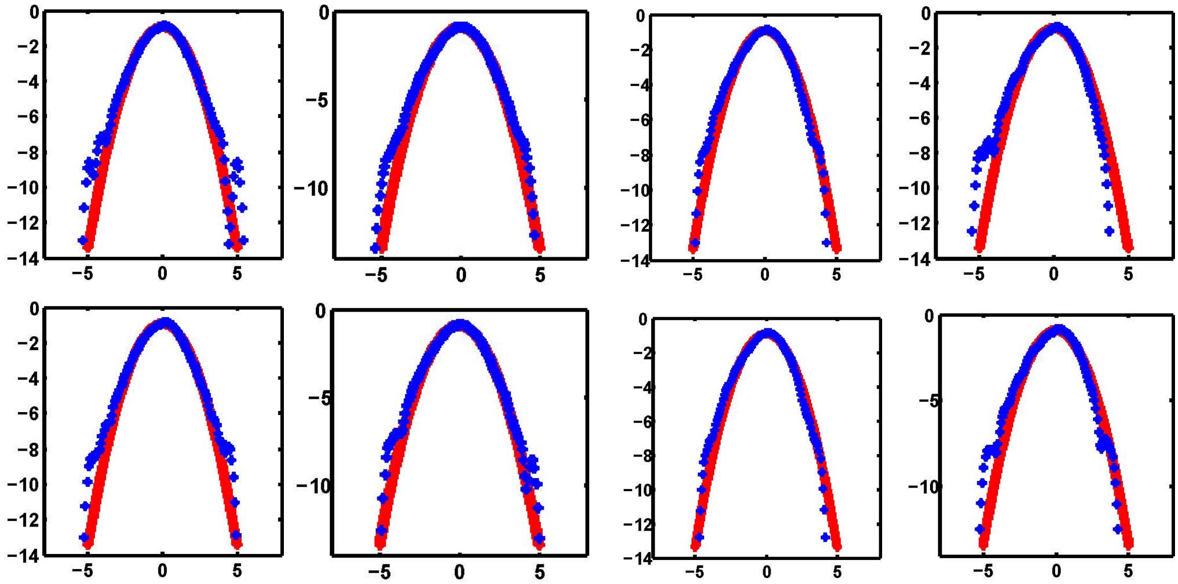

Figure 8: Log of Kernel smoothing density estimate vs Log of Normal Kernel for $\frac{\hat{\varepsilon}_t}{\sigma_{t,LLR}}$ (upper) and $\frac{\hat{\varepsilon}_t}{\sigma_{t,FTSG}}$ (lower) of Tokyo (left), Osaka (left middle), Beijing (right middle), Taipei (right)

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City	Code	$F_{C24AT Bloomberg}$	$F_{C24AT,\hat{\theta}_t^0}$	$F_{C24AT,\hat{\theta}_t^i}$	$F_{C24AT,\hat{\theta}_t^{spl}}$
Tokyo	J9	450.000	452.125	448.124	461.213
	K9	592.000	630.895	592.000	640.744
Osaka	J9	460.000	456.498	459.149	-
	K9	627.000	663.823	624.762	-

Table 9: Tokyo & Osaka C24AT future prices estimates on 20090130 from different MPR parametrization methods.

to the number of traded contracts (7). One can notice that the C24AT index futures for Tokyo are underpriced when the MPR is equal to zero. From (21) for C24AT index futures, we observe that a high proportion of the price value comes from the seasonal exposure, showing high CAT temperature futures prices from June to August and low prices from November to February. The influence of the temperature variation σ_t can be clearly reflected in the behaviour of the MPR. For both parametrization, MPR is close to zero far from measurement period and it jumps when it is getting closed to it. This phenomena is also related to the Samuelson effect, where the CAT volatility for each contract is getting closed to zero when the time to measurement period is large. C24AT index futures future prices with constant MPR estimate per contract per trading day full replicate the Bloomberg estimates and pricing deviations are smoothed over time when the estimations use smoothed MPRs. Positive (negative) MPR contributes positively (negatively) to future prices, leading to larger (smaller) estimation values than the real prices.

The Chicago Mercantile exchange does not carry out trade CDD futures for Asia, however one can use the estimates of the smoothed MPR of CAT (C24AT) futures in (21) to price CDD futures. From the HDD-CDD parity (5), one can estimate HDD futures and compare them with real data.

Since C24AT futures are indeed tradable assets, a simple and sufficient parametrization of the MPR to make the discounted asset prices martingales is setting $\theta_t = (\mu_t - r_t)/\sigma_t$. In order to see which of the components ($\mu_t - r_t$ or σ_t) contributes more to the variation of the MPR,

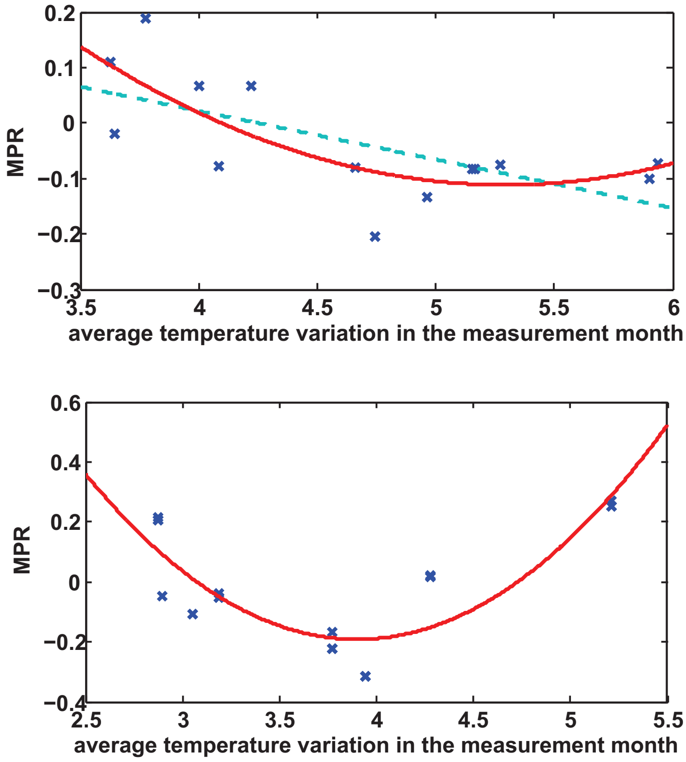

Figure 9: Average of the Calibrated MPR and the Temperature Variation of CAT-C24AT Futures with Measurement Period (MP) in 1 month (Linear, quadratic). Berlin and Essen (left) and Tokyo (right) from July 2008 to June 2009.

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the seasonal effect that the MPR θ_t presents was related with the seasonal variation σ_t of the underlying process. In this case, the relationship between θ_t and σ_t is well defined given by the deterministic form of $\sigma_t(\sigma_{t,FTSG}, \sigma_{t,LLR})$ in the temperature process.

First, using different trading day samples, the average of the calibrated θ_t^i over the period $[\tau_1, \tau_2]$ was estimated as:

$$\hat{\theta}^i_{[\tau_1,\tau_2]} = \frac{1}{T_{\tau_1,\tau_2} - t_{\tau_1,\tau_2}} \sum_{t=t_{\tau_1,\tau_2}}^{T_{\tau_1,\tau_2}} \hat{\theta}^i_t,$$

where $t_{[\tau_1,\tau_2]}$ and $T_{[\tau_1,\tau_2]}$ indicate the first and the last trade for the contracts with measurement month $[\tau_1, \tau_2]$. Similarly, the variation over the measurement period $[\tau_1, \tau_2]$ was defined as:

$$\hat{\sigma}^2_{[\tau_1,\tau_2]} = \frac{1}{\tau_2 - \tau_1} \sum_{t=\tau_1}^{\tau_2} \hat{\sigma}^2_t.$$

Then one can conduct a regression model of $\hat{\theta}^i_{\tau_1,\tau_2}$ on $\hat{\sigma}^2_{\tau_1,\tau_2}$. Figure 9 shows the linear and quadratic regression of the average of the calibrated MPR and $\sigma_t(\sigma_{t,FTSG}, \sigma_{t,LLR})$ of CAT-C24AT Futures with Measurement Period (MP) in 1 month for Berlin-Essen and Tokyo weather derivative from July 2008 to June 2009. The values of $\hat{\theta}^i_t$ for contracts on Berlin and Essen were assumed coming from the same population, while for the asian temperature market, Tokyo was the only considered one for being the largest one. As we expect, the contribution of σ_t into $\theta_t = (\mu_t - r_t)/\sigma_t$

City	Parameters	$\hat{\theta}_{\tau_1,\tau_2} = a + b \cdot \hat{\sigma}^2_{\tau_1,\tau_2}$	$\hat{\theta}_{\tau_1,\tau_2} = a + b \cdot \hat{\sigma}^2_{\tau_1,\tau_2} + c \cdot \hat{\sigma}^4_{\tau_1,\tau_2}$
Berlin-	a	0.3714	2.0640
Essen	b	-0.0874	-0.8215
	c	-	0.0776
	R^2_{adj}	0.4157	0.4902
Tokyo	a	-	4.08
	b	-	-2.19
	c	-	0.28
	R^2_{adj}	-	0.71

Table 10: Parametrization of MPR in terms of seasonal variation for contracts with measurement period of 1 month.

i	1	2	3	4	5	6
$\hat{c}_i$	5.11	-1.34	-0.39	0.61	0.56	0.34
$\hat{d}_i$	-162.64	19.56	16.72	28.86	16.63	21.84

Table 11: Coefficients of the seasonal function with trend for Koahsiung

gets larger the closer the contracts are to the measurement period. Table 10 shows the coefficients of the parametrization of $\hat{\theta}^i_t$ for the German and Japanese temperature market. A quadratic regression was fitting more suitable than a linear regression (see R^2 coefficients).

The previous findings generally support theoretical results of reverse relation between MPR $\hat{\theta}_{\tau_1,\tau_2}$ and seasonal variation $\sigma_t(\sigma_{t,FTSG}, \sigma_{t,LLR})$, indicating that a simple parametrization is possible. Therefore, the MPR for regions without weather derivative markets can be infered by calibration of the data or by knowing the formal dependence of MPR on seasonal variation. We conducted an empirical analysis to weather data in Koahsiung, which is located in the south of China and it is characterized by not having a formal temperature market, see Figure 10. In a similar way that other Asian cities, a seasonal function with trend was fitted:

$$\begin{aligned} \hat{\Lambda}_t &= 24.4 + 16 \cdot 10^{-5} t + \sum_{i=1}^{3} \hat{c}_i \cdot \cos\left\{\frac{2\pi i(t-\hat{d}_i)}{365}\right\} \\ &+ I(t \in \omega) \cdot \sum_{i=4}^{6} \hat{c}_i \cdot \cos\left\{\frac{2\pi (i-4)(t-\hat{d}_i)}{365}\right\} \end{aligned} \tag{37}$$

where $I(t \in \omega)$ is the indicator function for the months of December, January and February. This form of the seasonal function makes possible to capture the peaks of the temperature in Koahsiung, see upper panel of Figure 10. The coefficient values of the fitted seasonal function are shown in Table 11.

The fitted AR(p) process to the residuals of Koahsiung by AIC was of degree $p = 3$, where

$$\beta_1 = 0.77, \beta_2 = -0.12, \beta_3 = 0.04$$

and $CAR(3)$ with coefficients

$$\alpha_1 = -2.24, \alpha_2 = -1.59, \alpha_3 = -0.31$$

The seasonal volatility fitted with Local Linear Regression (LLR) is plotted in the middle panel of Figure 10, showing high volatility in late winter - late spring and low volatility in early summer - early winter. The standardized residuals after removing the seasonal volatility are very closed to normality (kurtosis=3.22, skewness=-0.08, JB=128.74), see lower panel of Figure 10.

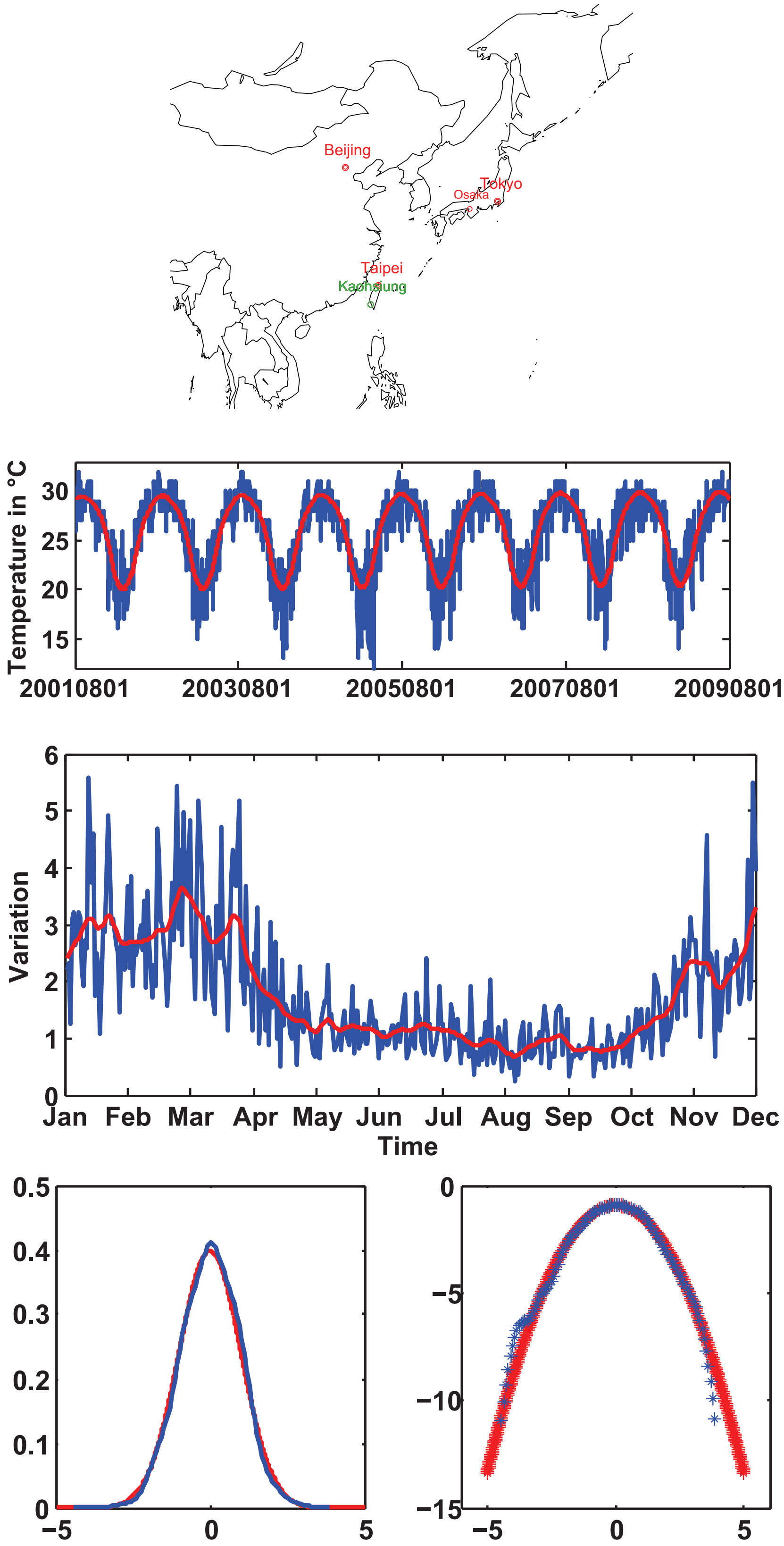

Figure 10: Map, Seasonal function with trend (upper), Seasonal volatility function (middle) and Kernel smoothing density estimate vs Normal Kernel for $\frac{\hat{\varepsilon}_t}{\sigma_{t,LLR}}$ (lower) for Koahsiung

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Derivative Type	Parameters
Index	C24AT
r	1%
t	1. September 2009
Measurement Period	27-31. October 2009
Strike	125°C
Tick Value	0.01°C=¥25
$F_{C24AT(20090901,20091027,20091031)}$	139.60
$C_{C24AT(20090901,20090908,20091027,20091031)}$	12.25
$C_{C24AT(20090901,20090915,20091027,20091031)}$	10.29
$C_{C24AT(20090901,20090922,20091027,20091031)}$	8.69
$C_{C24AT(20090901,20090929,20091027,20091031)}$	7.25

Table 12: C24AT Calls in Koahsiung

For $0 \leq t \leq \tau_1 < \tau_2$, the C24AT Future Contract for Kaohsiung is equal to:

$$\begin{aligned}
F_{C24AT(t,\tau_1,\tau_2)} &= \mathrm{E}^{Q_\theta}\left[\int_{\tau_1}^{\tau_2} T_s ds|\mathcal{F}_t\right] \\
&= \int_{\tau_1}^{\tau_2} \Lambda_u du + \mathbf{a}_{t,\tau_1,\tau_2}\mathbf{X}_t + \int_t^{\tau_1} \hat{\theta}_{\tau_1,\tau_2}\sigma_u \mathbf{a}_{t,\tau_1,\tau_2}\mathbf{e}_p du \\
&+ \int_{\tau_1}^{\tau_2} \hat{\theta}_{\tau_1,\tau_2}\sigma_u \mathbf{e}_1^\top A^{-1}\left[\exp\{A(\tau_2-u)\} - I_p\right]\mathbf{e}_p du
\end{aligned} \tag{38}$$

where $\hat{\theta}_{\tau_1,\tau_2} = 4.08 - 2.19 \cdot \hat{\sigma}^2_{\tau_1,\tau_2} + 0.028 \cdot \hat{\sigma}^4_{\tau_1,\tau_2}$, i.e. the formal dependence of MPR on seasonal variation for C24AT-Tokyo futures. In this case $\hat{\sigma}^2_{\tau_1,\tau_2}$=1.10, $\hat{\theta}_{\tau_1,\tau_2}$=2.01 and $F_{C24AT(20090901,20091027,20091031)}$=139.60.

The C24AT-Call Option written on a C24AT future with strike K at exercise time $\tau < \tau_1$ during period $[\tau_1, \tau_2]$ is equal to:

$$\begin{aligned}
C_{C24AT(t,\tau,\tau_1,\tau_2)} = &\exp\{-r(\tau-t)\} \\
&\times \Big[\left(F_{C24AT(t,\tau_1,\tau_2)} - K\right)\Phi\{d(t,\tau,\tau_1,\tau_2)\} \\
&+ \int_t^\tau \Sigma^2_{C24AT(s,\tau_1,\tau_2)} ds \Phi\{d(t,\tau,\tau_1,\tau_2)\}\Big],
\end{aligned}$$

Table 12 shows the value of the C24AT-Call Option written on a C24AT future with strike price $K = 125°$C, the measurement period during the 27-31th October 2009 and trading date on 1st. September 2009. The price of the C24AT-Call for Koahsiung decreases when the measurement period is getting closer. This example give us the insight that by knowing the formal dependence of MPR on seasonal variation, one can infere the MPR for regions where weather derivative market does not exist and with that one can price new exotic derivatives. Without doubt, the empirical findings of the MPR need to be further developed to better understand its behaviour.

6 Conclusion

This paper analyses the pricing of asian weather risk. We apply higher order continuous time autoregressive models CAR(3) with seasonal variance for modelling Asian temperature. We modelled the seasonal variation with a GARCH model and with a local linear regression in order to achieve normal residuals and with that being able to work in a financial mathematics context.

From temperature derivative (C24AT) data of the Chicago Mercantile Exchange (CME), the calibration of the market price of risk is estimated to price new weather derivatives. The MPR for C24AAT temperature derivatives is different from zero, showing a seasonal structure that comes from the seasonal variance of the temperature process. The empirical findings in this paper generally support theoretical results of reverse relation between MPR and variation. Therefore, by knowing the formal dependence of MPR on seasonal variation, one can infere the MPR for regions where weather derivative market does not exist.

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